This is a preprint of a paper published in:
International Journal of Shape Modeling 8(1):23-42, Jun 2002

On the geometry of recursive subdivision

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June 15, 2001

Abstract

In this paper we investigate the properties of recursive subdivision from a geometric point of view. We explore the connections between subdivision and such areas of mathematics as spherical trigonometry, inversive geometry, and orthogonal polynomials. The methods we develop give new insights to well-known subdivision schemes and they can also be used in the argued construction of new schemes with prescribed properties.

1 Introduction

In recursive subdivision, we start with an initial mesh of vertices and edges, and in each step, we insert new vertices, calculated as linear combinations of the existing ones, and we connect them with edges, producing a refined mesh. In approximating schemes, in each step, we also adjust the old vertices, again as a linear combination of the existing ones. A subdivision scheme is called stationary if the weights of the linear combinations are the same in each step. The information about these linear combinations is usually coded in masks, that is planar configurations of vertices and edges, with the weights of the linear combinations adjusted on the vertices. Usually, these masks are highly symmetric, because the linearity of the subdivision process means that the surface is invariant under an affine transformation of the set of initial points, and in particular under reflections and rotations of that set.

Two basic questions naturally arising in the study of subdivision schemes are: what weights should we choose for the linear combinations to optimise in some way the properties of the scheme, and what are the properties of a scheme

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given a choice of weights. The main analytic tools to tackle these questions are the eigenanalysis of the subdivision matrix, see [3], the generating function formalism, see [4], and the characteristic map, see [8].

In this paper we will take a more geometric approach. Instead of working on a planar mask, which can be also seen as the parametric space, we work with symmetric configurations of points lying on other than the plane surfaces. By taking linear combinations of these points we try to determine new points satisfying some particular geometric considerations. For interpolating schemes the standard requirement for the new points is to lie on the same surface as the initial configuration. In a variation of this approach we can start with an initial symmetric planar configuration, apply a transformation that sends the plane on a surface of our choice, and then work with this new configuration.

Here, most of the time, that underlying surface will be the sphere. One reason for that choice is the simplicity of the sphere that makes the calculations easier. Another is that locally all the points of a sphere look alike, and the only other surfaces with that property are the plane and the cylinder. A third reason is that the inverse of the square of the radius of the sphere can be seen as a local estimator of the Gaussian curvature. Nevertheless, sometimes we will work on the hyperboloid and we will see that most of the results are identical. As the only other simply connected surface with constant curvature, that is the Euclidean plane, gives trivial schemes, our treatment, in a sense, is complete.

In the univariate case the equivalent of working on a sphere is working on a circle. Sometimes we will use the word sphere irrespective of the dimension to make the text lighter.

2 The geometry of recursive subdivision

Working on a sphere, a first approach to the construction of a subdivision scheme is to start with a symmetric configuration of points on that sphere, and then calculate explicitly the new point as a linear combination of that symmetric configuration. Such a method, as we expect, involves a lot of trigonometry and the difficulty of the calculations increases rapidly with the complexity of the problem, but gives exact, from our point of view, results.

A second approach is to start with a symmetric planar configuration, project it on the sphere, and then calculate the new points as a linear combination of the points on the projected image. This approach, as we expect, involves a lot of inversive geometry, and is not so exact, in the sense that the transformation we apply is not an isometry and thus the projected configuration on the sphere is not as symmetric as we wish. Nevertheless, because the transformation is conformal, as the subdivision step increases, we expect the two methods to give the same result in the limit. Moreover, with the second method we have to handle rational functions, rather than trigonometric, making the algebraic and analytic manipulation much easier.

2.1 Spherical symmetric configuration

To calculate a point on the unit sphere as a linear combination, with sum of weights 1, of some other points of the sphere, does not pose any theoretical difficulties. A first observation we can make is that not all the weights can be positive. Indeed, in that case, if O is the origin, and so the centre of the unit sphere, the equation

$$\alpha_1 \vec{OP_1} + \alpha_2 \vec{OP_2} + \cdots + \alpha_n \vec{OP_n} = \vec{OP} \tag{1}$$

gives

$$|\alpha_1||\vec{OP_1}| + |\alpha_2||\vec{OP_2}| + \cdots + |\alpha_n||\vec{OP_n}| \geq |\vec{OP}| \tag{2}$$

Also, $|\vec{OP_i}| = 1$ for $i = 1, 2, \ldots, n$, giving

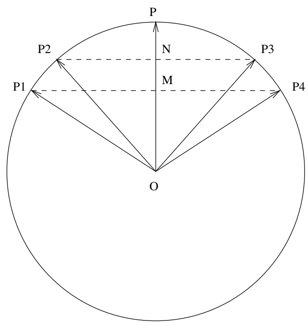
$$|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| \geq 1 \tag{3}$$

with equality holding if and only if all the vectors are colinear. If all the α_i, $i = 1, 2, \ldots, n$, are positive, then (3) holds as an equation. That means

$$\vec{OP_1} = \vec{OP_2} = \cdots = \vec{OP_n} \tag{4}$$

giving $P_1 = P_2 = \cdots = P_n$.

So, if all the coefficients are positive the new point P lie in the interior of the sphere, and we have to find a way to use negative coefficients to increase the distance of the new point from the origin. Indeed, suppose we found two linear combinations L_1, L_2 of the initial points, with sum of weights 1, giving the points M, N respectively. Suppose that M, N lie on the line OP, and $M \neq N$. The figure [1] shows a simple case for dimension 1.



We have

$$\vec{OM} = \frac{1}{2}(\vec{OP_1} + \vec{OP_4}) = \mu\vec{OP}, \quad \vec{ON} = \frac{1}{2}(\vec{OP_1} + \vec{OP_4}) = \nu\vec{OP}, \quad \mu \neq \nu \tag{5}$$

So, the linear combination

$$\vec{ON} + \alpha(\vec{ON} - \vec{OM}) = (\nu + \alpha(\nu - \mu))\vec{OP} \tag{6}$$

where α is a variable, gives all the points on the line OP, and has sum of weights 1 as required. We just have to calculate the value of α that will give the point P.

In general, if $P_1, P_2, \ldots P_k$ are points on a sphere, we take n linear combinations

$$L_1, L_2, \ldots, L_n$$

each one with sum of weights 1, such that the new points

$$L_1(P_1, \ldots, P_k), L_2(P_1, \ldots, P_k), \ldots, L_n(P_1, \ldots, P_k)$$

are colinear. Using $n-1$ variables we form a linear combination like,

$$L_1 + \alpha_1(L_1 - L_2) + \alpha_2(L_2 - L_3) + \cdots + \alpha_{n-1}(L_{n-1} - L_n) \tag{7}$$

or,

$$L_1 + \alpha_1(L_1 - L_2) + \alpha_2(L_1 - L_3) + \cdots + \alpha_{n-1}(L_1 - L_n) \tag{8}$$

that by construction guarantees sum of weights 1. Then we solve for the $n-1$ variables $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ to satisfy certain requirements.

Some remarks that will help understanding how we can use the above procedure

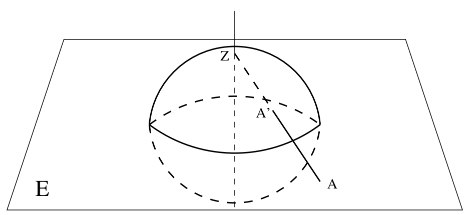
(i) The values of $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ depend on the initial configuration of the points $P_1, P_2, \ldots P_k$ on the sphere. Requiring that configuration to have the highest possible symmetry usually determines it up to similarity. In that case, the positions of $P_1, P_2, \ldots, P_k$ are functions of one variable. Usually we will use a variable θ corresponding to arcs on the sphere. The initial value of θ, before the first subdivision step, is strongly connected to the curvature as it determines the radius of the sphere the initial data lie on. We can use this variable as a curvature controlling tension parameter.

(ii) As we perform more steps of the subdivision process the size of the mask shrinks, and in the limit becomes 0. That is, in the limit we get a stationary scheme.

(iii) In particular, we can resolve artifacts by using the varied coefficients in the first few steps of the subdivision because we know that these force the new points, in the symmetric case, to lie on the sphere.

2.2 Transformation of a planar symmetric configuration

In this second approach we start with a symmetric configuration on the plane, and we project it on the unit sphere. That is, we project every point of the plane $z = 0$ towards or away from the point $Z = (0, 0, 1)$ until it meets the unit sphere. See figure [2].



This projection is equivalent to a reflection in the sphere with centre (0,0,1) and radius $\sqrt{2}$, and so is a continuous conformal transformation. See for example [7].

If $A=(x,y,0)$ is a point on the plane $z=0$ and $a=\sqrt{x^2+y^2}$ is its distance from the origin then the image of A is

$$A' = (\frac{2x}{1+a^2}, \frac{2y}{1+a^2}, \frac{a^2-1}{1+a^2}) \tag{9}$$

We notice that the neighbourhood of O maps to a neighbourhood of (0,0,-1) on the sphere, so if we prefer to work with positive numbers we can also apply a reflection through the plane $z=0$, that is, to work with the inversion

$$A \to \sigma(A) = (\frac{2x}{1+a^2}, \frac{2y}{1+a^2}, \frac{1-a^2}{1+a^2}) \tag{10}$$

Let also

$$\sigma_z(A) = \frac{1-a^2}{1+a^2} \tag{11}$$

denote the z-coordinate of $\sigma(A)$.

Working as in the case of spherical symmetric configuration, we start with a set of points $P_1, P_2, \ldots, P_k$ on the plane forming a configuration symmetric around the origin O of the plane. We find linear combinations

$$L_1, L_2, \ldots, L_n$$

with sum of weights 1, satisfying

$$L_i(P_1, \ldots, P_k) = O \qquad i = 1, \ldots, n \tag{12}$$

To keep the formulae as simple as possible, and without much loss of generality, the non-zero coefficients will correspond to points equidistant from O. Then we form a linear combination of L_i's, again with sum of weights 1, like

$$L = L_1 + \alpha_1(L_1 - L_2) + \cdots + \alpha_{n-1}(L_1 - L_n) \tag{13}$$

and we calculate the α_i to optimise the resulting scheme.

As we said above this method is not as accurate as that of the previous section, so the optimisation will be mainly relevant to the limit case. We notice that as we proceed with the subdivision process all the points P_i converge to O. So a first necessary requirement for a continuous scheme is that

$$L(\sigma_z(P_1), \ldots, \sigma_z(P_k)) = \sigma_z(O) = 1 \tag{14}$$

To continue, we express all the distances of the initial points P_i from O as multiples of a distance a. Let $c_i a$ be the distance from O of the points of L_i. Then

$$L_i(\sigma_z(P_1), \ldots, \sigma_z(P_k)) = \frac{1-c_i^2a^2}{1+c_i^2a^2} \qquad i = 1, \ldots, n \tag{15}$$

and (14) gives

$$\frac{1-c_1^2a^2}{1+c_1^2a^2} + \alpha_1(\frac{1-c_1^2a^2}{1+c_1^2a^2} - \frac{1-c_2^2a^2}{1+c_2^2a^2}) + \cdots + \alpha_{n-1}(\frac{1-c_1^2a^2}{1+c_1^2a^2} - \frac{1-c_{n-1}^2a^2}{1+c_{n-1}^2a^2}) - 1 = 0 \tag{16}$$

giving,

$$\frac{-2c_1^2a^2}{1+c_1^2a^2} + \alpha_1\frac{2(c_2^2-c_1^2)a^2}{(1+c_1^2a^2)(1+c_2^2a^2)} + \cdots + \alpha_{n-1}\frac{2(c_n^2-c_1^2)a^2}{(1+c_1^2a^2)(1+c_n^2a^2)} = 0 \tag{17}$$

After summing up the fractions we get

$$\frac{P(a^2)}{Q(a^2)} = 0 \tag{18}$$

where $Q(a^2)$ has the form

$$Q(a^2) = (1 + c_1^2 a^2)(1 + c_2 a^2) \cdots (1 + c_n a^2) \tag{19}$$

and the constant coefficient of $P(a^2)$ is 0. That means that

$$\lim_{a \to 0} \frac{P(a^2)}{Q(a^2)} = 0 \tag{20}$$

and the scheme is continuous for any choice of $\alpha_1, \ldots \alpha_{n-1}$.

So, we will determine $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ to optimise the smoothness of the resulting scheme. Let A be a point of distance a from O, then

$$\Delta\sigma_z(0, A) = \sigma_z(0) - \sigma_z(A) = 1 - \frac{1 - a^2}{1 + a^2} = \frac{2a^2}{1 + a^2} \tag{21}$$

and so,

$$\frac{P(a^2)}{Q(a^2)\Delta\sigma_z(0, A)} \tag{22}$$

has a limit 0 for $A \to O$, if and only if the term a^2 of $P(a^2)$ also vanishes. In general, if the smallest non-vanishing term of $P(a^2)$ is a^{2k}, then

$$\lim_{a \to 0} \frac{P(a^2)}{Q(a^2)(\Delta\sigma_z(0, A))^{k-1}} = 0 \tag{23}$$

and we will calculate the $\alpha_1, \ldots \alpha_{n-1}$ such that the first non-vanishing power of a^2 is as large as possible. We have to solve a $(n-1) \times (n-1)$ homogeneous system, and if it is not degenerate we will find unique α_i's such that the smallest term of $P(a^2)$ is a^{2n}. From an analytical point of view, that means that the first $n-1$ derivatives of P vanish at 0.

The univariate case is similar. The projection is on the unit circle rather than the unit sphere, and the reflection through the x-axis.

Remark 1: If σ' is the standard projection of the initial planar mask to the positive sheet of the hyperboloid

$$z^2 - (x^2 + y^2) = 1 \tag{24}$$

we have the equivalent of equation (11), see [7]

$$\sigma'_z(A) = \frac{1 + a^2}{1 - a^2} \tag{25}$$

So, in (18) instead of a polynomial

$$P(a^2) = x_1 a^2 + x_2 a^4 + \cdots + x_n a^{2n} \tag{26}$$

we have the polynomial

$$P(-a^2) = -x_1 a^2 + x_2 a^4 + \cdots + (-1)^n x_n a^{2n} \tag{27}$$

and instead of the linear system

$$x_1 = x_2 = \cdots = x_{n-1} = 0 \tag{28}$$

we solve the system

$$-x_1 = x_2 = \cdots = (-1)^{n-1} x_{n-1} = 0 \tag{29}$$

Clearly the two systems are equivalent and result the same subdivision coefficients.

We can also work directly on symmetric configurations on the hyperboloid as we did in the previous section on the sphere. This will lead us to the same equations but with hyperbolic trigonometric functions rather than trigonometric. For all our purposes the calculus of hyperbolic trigonometric functions is identical with the calculus of trigonometric functions, and again these two approaches will give in the limit the same result.

Remark 2: The subdivision coefficients depend on the distances $c_i a$ and the transformation of the plane, which here is the stereographic projection on the sphere or the hyperboloid. Clearly we can also work conversely and for a given set of coefficients find the implied distances c_i. That is, a metric on the plane gives rise to a set of subdivision coefficients and vice-versa. It is conceivable that we can express the ideas of this section just in terms of metric spaces and their transformations.

Remark 3: In a slightly more general but less instructive approach, we can use n variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ to form the linear combination

$$L = \alpha_1 L_1 + \alpha_2 L_2 + \cdots + \alpha_n L_n \tag{30}$$

Then the requirement for a continuous scheme would give the relation

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1 \tag{31}$$

3 Examples

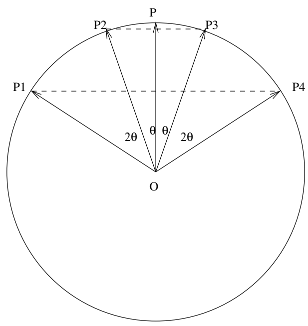
In this section we will give several examples to illustrate the methods described above. We give a variety of examples highlighting all the points made above. So, we give univariate as well as bivariate examples, examples giving well known schemes and examples giving new schemes and we use alternately trigonometry and inversive geometry.

3.1 4-point scheme

Suppose that we have 4 points P_1, P_2, P_3, P_4 lying on a circle of centre O and radius 1, such that

$$\widehat{P_1OP_2} = \widehat{P_2OP_3} = \widehat{P_3OP_4} = 2\theta \tag{32}$$

See figure [3].



To calculate the midpoint P of the arc P_2P_3 as a linear combination with sum of weights 1 of these points we start with the linear combinations

$$L_1 = \frac{1}{2}(\vec{OP_2} + \vec{OP_3}) \qquad L_2 = \frac{1}{2}(\vec{OP_1} + \vec{OP_4}) \tag{33}$$

We want

$$L_1 + \alpha_1(L_1 - L_2) = \frac{1}{2}(\vec{OP_2} + \vec{OP_3}) + \alpha_1 \frac{1}{2}(\vec{OP_2} + \vec{OP_3} - \vec{OP_1} - \vec{OP_4}) = \vec{OP} \tag{34}$$

and taking the norms of the colinear vectors corresponding to L_1, L_2 we have

$$\cos\theta + \alpha_1(\cos\theta - \cos 3\theta) = 1 \tag{35}$$

giving

$$\begin{aligned} \alpha_1 &= \frac{1 - \cos\theta}{\cos\theta - \cos 3\theta} = \frac{1 - \cos\theta}{\cos\theta - (4\cos^3\theta - 3\cos\theta)} = \\ &= \frac{1 - \cos\theta}{4\cos\theta(1 + \cos\theta)(1 - \cos\theta)} = \frac{1}{4\cos\theta(1 + \cos\theta)} \end{aligned} \tag{36}$$

That means,

$$\lim_{\theta \to 0} \alpha_1 = \frac{1}{8} \tag{37}$$

and so, in the limit, the mask of the scheme is

$$-\frac{1}{16}, \frac{9}{16}, \frac{9}{16}, -\frac{1}{16}$$

that is the 4-point scheme in [5].

As we mentioned above, we can see the initial value of the angle θ as a variable. At the end of the paper we draw the curve interpolating the corners of a square for several initial values of that variable θ, and halving it at each step, showing that we can use it as a tension parameter. Then we do the same substituting $\cosh\theta$ for $\cos\theta$ in (36). Notice that the intervals of interest are

$$0 \le \theta < \frac{\pi}{2} \quad \text{giving} \quad \frac{1}{8} \le \alpha_1 < \infty \tag{38}$$

and

$$0 \le \theta < \infty \quad \text{giving} \quad \frac{1}{8} \ge \alpha_1 > 0 \tag{39}$$

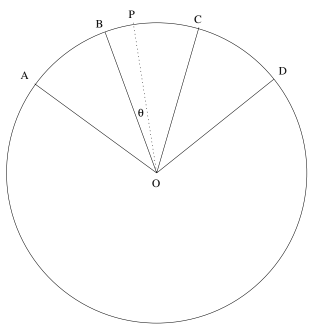
correspondingly.

3.2 4-point n-ary scheme

Suppose we have 4 points A, B, C, D equally distanced on a circle of centre O. We want to calculate a new point P on the circle, as a linear combination of A, B, C, D with sum of weights 1, such that

$$\widehat{BOP} = \frac{1}{n-1}\widehat{POC} = \theta \tag{40}$$

See figure [4]



We have,

$$\widehat{AOP} = (n+1)\theta \qquad \widehat{POC} = (n-1)\theta \qquad \widehat{POD} = (2n-1)\theta \tag{41}$$

We write the linear combination

$$(\frac{n-1}{n}OB + \frac{1}{n}OC) + \alpha_1[(\frac{n-1}{n}OB + \frac{1}{n}OC) - (\frac{2n-1}{3n}OA + \frac{n+1}{3n}OD)] \tag{42}$$

and working as in the case of the 4-point scheme we find

$$\alpha_1 = \lim_{\theta\to 0} \frac{1 - \frac{1}{n}[(n-1)\cos\theta + \cos(n-1)\theta]}{\frac{1}{n}[(n-1)\cos\theta + \cos(n-1)\theta] - \frac{1}{3n}[(2n-1)\cos(n+1)\theta + (n+1)\cos(2n-1)\theta]}$$

which can be written

$$\lim_{\theta\to 0} \frac{1 - \frac{1}{n}[(n-1)T_1(\cos\theta) + T_{n-1}(\cos\theta)]}{\frac{1}{n}[(n-1)T_1(\cos\theta) + T_{n-1}(\cos\theta)] - \frac{1}{3n}[(2n-1)T_{n+1}(\cos\theta) + (n+1)T_{2n-1}(\cos\theta)]}$$

with T_n the Chebyshev polynomial of first kind. As cos0=1, the limit is equal to

$$\lim_{x\to 1}\frac{1-\frac{1}{n}[(n-1)T_1(x)+T_{n-1}(x)]}{\frac{1}{n}[(n-1)T_1(x)+T_{n-1}(x)]-\frac{1}{3n}[(2n-1)T_{n+1}(x)+(n+1)T_{2n-1}(x)]}$$

To calculate the limit we will use the following basic facts from the theory of Chebyshev polynomials, see for example [1].

$$T_n(1)=1 \tag{43}$$

$$U_n(1)=n+1 \tag{44}$$

$$T'_n(x)=nU_{n-1}(x) \tag{45}$$

where U_n are the Chebyshev polynomials of the second kind.

Using (43) we see that both numerator and denominator tend to 0 as $x\to 1$. By L'Hospital rule we can differentiate both numerator and denominator without changing the limit. Using (45), and writing U_n instead of $U_n(x)$, we get

$$\lim_{x\to 1}\frac{-\frac{1}{n}[(n-1)U_0+(n-1)U_{n-2}]}{\frac{1}{n}[(n-1)U_0+(n-1)U_{n-2}]-\frac{1}{3n}[(n+1)(2n-1)U_n+(n+1)(2n-1)U_{2n-2}]}$$

which, using (44), gives

$$\frac{-\frac{1}{n}[(n-1)+(n-1)^2]}{\frac{1}{n}[(n-1)+(n-1)^2]-\frac{1}{3n}[(n+1)^2(2n-1)+(n+1)(2n-1)^2]}$$

which is equal to,

$$\frac{-\frac{1}{n}n(n-1)}{\frac{1}{n}n(n-1)-\frac{1}{3n}(n+1)(2n-1)(3n)}=\frac{-(n-1)}{n-1-(2n^2+n-1)}=\frac{n-1}{2n^2}$$

For example, if $n=3$ we get the mask

$$\frac{-5}{81},\frac{20}{27},\frac{10}{27},\frac{-4}{81}$$

that is, we have the 4-point ternary scheme described in [2].

For $n>3$ we also need to consider

$$\widehat{BOP}=\frac{k}{n-1}\widehat{POC}=k\theta,\quad k=1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor \tag{46}$$

In that case we have

$$\widehat{AOP}=(n+k)\theta\qquad\widehat{POC}=(n-k)\theta\qquad\widehat{POD}=(2n-k)\theta \tag{47}$$

and working similarly to the case $k=1$, we have to calculate

$$\lim_{\theta\to 0}\frac{1-\frac{1}{n}[(n-k)\cos k\theta+k\cos(n-k)\theta]}{\frac{1}{n}[(n-k)\cos k\theta+k\cos(n-k)\theta]-\frac{1}{3n}[(2n-k)\cos(n+k)\theta+(n+k)\cos(2n-k)\theta]}$$

giving,

$$\lim_{x\to 1}\frac{-\frac{1}{n}[(n-k)kU_{k-1}+k(n-k)U_{n-k-1}]}{\frac{1}{n}[(n-k)kU_{k-1}+k(n-k)U_{n-k-1}]-\frac{1}{3n}[(n+k)(2n-k)U_{n+k-1}+(n+k)(2n-k)U_{2n-k-1}]}$$

which gives,

$$\frac{-\frac{1}{n}[k^2(n-k)+k(n-k)^2]}{\frac{1}{n}[k^2(n-k)+k(n-k)^2]-\frac{1}{3n}[(n+k)^2(2n-k)+(n+k)(2n-k)^2]}=\frac{k(n-k)}{2n^2}$$

3.3 6-point binary scheme

We give this example to illustrate the method using three linear combinations and solving for two variables. Let A, B, C, D, E, F be equally distanced points on the x-axis. The origin O is in the barycentre of the configuration, that is,

$$OC = OD = a \quad OB = OE = 3a \quad OA = OF = 5a \tag{48}$$

Under the standard stereographic projection on the unit circle, followed by a reflection through the x-axis, the y-coordinate of the image A', B', C', D', E', F' of these 6 points will be

$$\frac{1-25a^2}{1+25a^2}, \frac{1-9a^2}{1+9a^2}, \frac{1-a^2}{1+a^2}, \frac{1-a^2}{1+a^2}, \frac{1-9a^2}{1+9a^2}, \frac{1-25a^2}{1+25a^2}$$

respectively. With the notation of (13), we choose

$$L_1 = \frac{1}{2}(OC + OD) \quad L_2 = \frac{1}{2}(OB + OE) \quad L_1 = \frac{1}{2}(OA + OF) \tag{49}$$

From (16) we get the equation

$$\frac{1-a^2}{1+a^2} + \alpha_1\left(\frac{1-a^2}{1+a^2} - \frac{1-9a^2}{1+9a^2}\right) + \alpha_2\left(\frac{1-a^2}{1+a^2} - \frac{1-25a^2}{1+25a^2}\right) - 1 = 0 \tag{50}$$

which after the calculations gives

$$\frac{-2a^2(1+9a^2)(1+25a^2) + \alpha_1 16a^2(1+25a^2) + \alpha_2 48a^2(1+9a^2)}{(1+a^2)(1+9a^2)(1+25a^2)} = 0 \tag{51}$$

To make the coefficient of a^2 vanish we must have

$$-2 + 16\alpha_1 + 48\alpha_2 = 0 \tag{52}$$

while the coefficient of a^4 vanishes if

$$-68 + 400\alpha_1 + 432\alpha_2 = 0 \tag{53}$$

Solving the system we get

$$\alpha_1 = \frac{25}{128} \quad \text{and} \quad \alpha_2 = \frac{-3}{128} \tag{54}$$

and we get the mask

$$\frac{3}{256}, \frac{-25}{256}, \frac{75}{128}, \frac{75}{128}, \frac{-25}{256}, \frac{3}{256} \tag{55}$$

So, the scheme we get is the $\mathbf{C}^2$ scheme described in [2]

3.4 $\sqrt{3}$-scheme

The $\sqrt{3}$-scheme for triangular meshes was intoduced in [6]. In each step a new vertex is inserted at the barycentre of each triangle, and every old vertex is relaxed according to a linear combination of itself and its direct neighbours. For symmetry reasons, we assign equal weights to each neighbour of P. Hence, using the notation in [6], if P is a vertex of valency n and $P_0, P_1, \ldots, P_{n-1}$ the vertices adjacent to P, the new position of P is given by

$$P' = (1 - a_n)P + a_n \frac{1}{n} \sum_{i=0}^{n-1} P_i \tag{56}$$

To analyse the geometry of that scheme we suppose that $P, P_0, \ldots, P_{n-1}$ all lie on a sphere of centre O and radius 1. We also suppose that $P_0, \ldots, P_{n-1}$ are the vertices of a planar regular n-gon and P is the projection on the sphere of the centre of the n-gon. Let also $F_0, F_1, \ldots, F_{n-1}$ be the barycentres of the plane triangles $PP_0P_1, PP_1P_2, \ldots, PP_{n-1}P_0$ respectively.

We want to calculate a displacement P' of P such that $P', F_0 \ldots, F_{n-1}$ also lie on a sphere, either on a sphere of radius 1 but with different centre O', or on a sphere with centre O and radius $r < 1$.

(i) In the first case we have a displacement of the sphere along the z-axis. We want to express it as a linear combination

$$\vec{OP'} = \vec{OP} - a_n(\vec{OP} - \frac{1}{n}(\vec{OP_0} + \vec{OP_1} + \cdots + \vec{OP_{n-1}})) \tag{57}$$

If

$$\theta = \widehat{POP_0} = \widehat{POP_1} = \cdots = \widehat{POP_{n-1}} \tag{58}$$

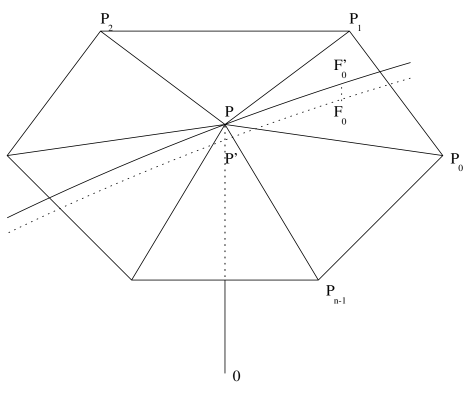
taking norms in equation (55) we have

$$|\vec{OP'} - \vec{OP}| = a_n(1 - \cos\theta) \tag{59}$$

giving

$$a_n = \frac{|P'P|}{1 - \cos\theta} \tag{60}$$

We have $|PP'| = |F_0F_0'|$ where F_0' is the intersection of the initial sphere with the line parallel to the z-axis passing through F_0. See figure [5].



We will calculate $|F_0F_0'|$. For convenience we write $\omega = \frac{2\pi}{n}$. In spherical coordinates we have

$$P = (1, 0, 0) \quad P_0 = (1, \theta, 0) \quad P_1 = (1, \theta, \omega) \tag{61}$$

and so in Cartesian coordinates we get

$$P = (0, 0, 1) \quad P_0 = (\sin\theta, 0, \cos\theta) \quad P_1 = (\sin\theta\cos\omega, \sin\theta\sin\omega, \cos\theta) \tag{62}$$

That means that

$$F_0 = (\frac{\sin\theta(\cos\omega + 1)}{3}, \frac{\sin\theta\sin\omega}{3}, \frac{2\cos\theta + 1}{3}) \tag{63}$$

So, the z coordinate of F_0 is

$$\frac{2\cos\theta + 1}{3} \tag{64}$$

while from the equation of the unit sphere and the x, y coordinates of F_0 we have that the z-coordinate of F_0' is

$$\sqrt{1 - \frac{\sin^2\theta(\cos\omega + 1)^2 + \sin^2\theta\sin^2\omega}{9}} \tag{65}$$

That gives,

$$\begin{aligned} |F_0F_0'| &= \sqrt{1 - \frac{\sin^2\theta(\cos\omega + 1)^2 + \sin^2\theta\sin^2\omega}{9}} - \frac{2\cos\theta + 1}{3} = \\ &= \frac{\sqrt{9 - 2\sin^2\theta(\cos\omega + 1)} - (2\cos\theta + 1)}{3} \end{aligned} \tag{66}$$

and finally (58) gives,

$$a_n = \frac{\sqrt{9 - 2\sin^2\theta(\cos\omega + 1)} - (2\cos\theta + 1)}{3(1 - \cos\theta)} \tag{67}$$

We can easily calculate that

$$\lim_{\theta \to 0} a_n = \frac{4 - 2\cos\omega}{9} \tag{68}$$

and so our analysis gives the same result as the eigenanalysis in [6].

(ii) In the second case where $P', F_0, \ldots, F_{n-1}$ lie on the same sphere of centre O, we have

$$|P'P| = 1 - |OF_0| = 1 - \sqrt{\frac{\sin^2\theta(\cos\omega + 1)^2 + \sin^2\theta\sin^2\omega + (2\cos\theta + 1)^2}{9}} \tag{69}$$

and

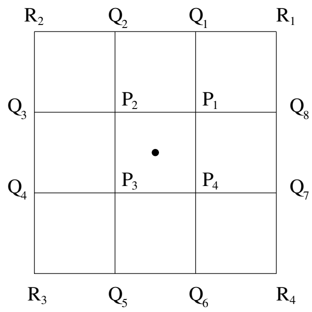
$$\alpha_n = \frac{1 - \sqrt{\frac{\sin^2\theta(\cos\omega+1)^2 + \sin^2\theta\sin^2\omega + (2\cos\theta+1)^2}{9}}}{1 - \cos\theta} =$$

$$= \frac{1 - \sqrt{\frac{2\sin^2\theta(\cos\omega+1)+(2\cos\theta+1)^2}{9}}}{1-\cos\theta} \tag{70}$$

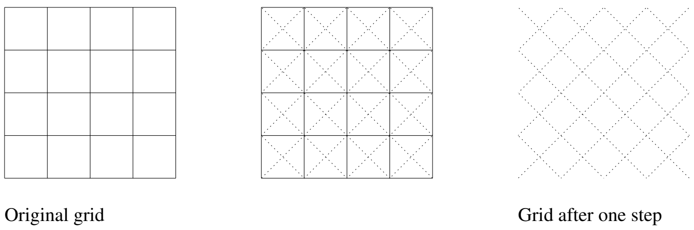
The limit as $\theta \to 0$ is the same as in the case (i), and so we have two different schemes with the same stationary limit, namely the $\sqrt{3}$-scheme. The second, with initial point data the vertices of a tetrahedron, octahedron, icosahedron, and initial angle equal to the arc corresponding to the edge of that regular solid, should produce a perfect sphere.

3.5 An interpolatory scheme for regular grids

As an example of how we can use the above methods to find new schemes we will describe an interpolatory $\sqrt{2}$-scheme. In each step a new point is inserted in the centre of each face, calculated as linear combination of its nearest 16-points. See figure [6].



Then the every new vertex is connected with its 4 nearest neighbours, while the original edges are removed causing a 45 degrees rotation of the grid. See figure [7].



Using the notation of section 2, we write

$$L_1 = \frac{1}{4}(P_1 + P_2 + P_3 + P_4) \tag{71}$$

$$L_2 = \frac{1}{8}(Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8) \tag{72}$$

$$L_3 = \frac{1}{4}(R_1 + R_2 + R_3 + R_4) \tag{73}$$

and we form the linear combination

$$L_1 + \alpha_1(L_1 - L_2) + \alpha_2(L_1 - L_3) \tag{74}$$

If a is half the edge of the one of the 9 squares of the mask if figure [6], the distances of the points P_i, Q_i, R_i from the new inserted point are $\sqrt{2}, \sqrt{10}, \sqrt{18}$ respectively. So, (16) gives the equation

$$\frac{1-2a^2}{1+2a^2} + \alpha_1\left(\frac{1-2a^2}{1+2a^2} - \frac{1-10a^2}{1+10a^2}\right) + \alpha_2\left(\frac{1-2a^2}{1+2a^2} - \frac{1-18a^2}{1+18a^2}\right) - 1 = 0 \tag{75}$$

giving,

$$\frac{-4a^2 - 112a^4 - 720a^6 + \alpha_1(16a^2 + 288a^4) + \alpha_2(32a^2 + 320a^4)}{(1+2a^2)(1+10a^2)(1+18a^2)} = 0 \tag{76}$$

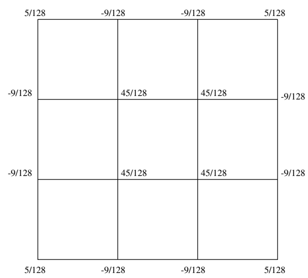
and we obtain the system

$$\begin{array}{llll} -4 & +16\alpha_1 & +32\alpha_2 & = 0 \\ -112 & +288\alpha_1 & +320\alpha_2 & = 0 \end{array}$$

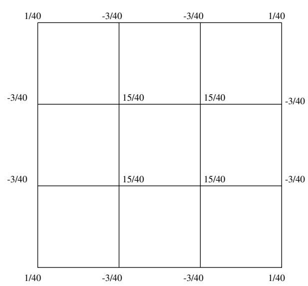
which has a solution

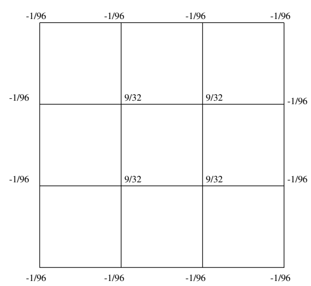
$$\alpha_1 = \frac{9}{16} \qquad \alpha_2 = \frac{-5}{32} \tag{77}$$

Hence, the coefficients of the scheme are $\frac{45}{128}, \frac{-9}{128}, \frac{5}{128}$ for the points P_i, Q_i, R_i, respectively.
Figure [8] shows the corresponding mask.



To show the relation between the subdivision coefficients and the metric of the parameter space we will calculate the subdivision schemes corresponding to the l_1, l_∞ metrics rather than the standard Euclidean l_2. In this case the distances between O and P_i, Q_i, R_i are 2,4,6, and 1,3,3 respectively. The corresponding masks will be, see figures [9],[10]





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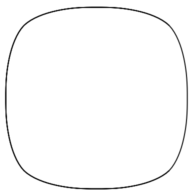
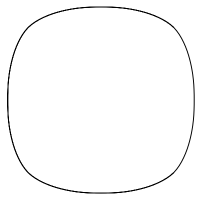
The interpolation of the vertices of a square with the 4-point binary scheme with spherical and hyperbolic geometry for several values of θ.

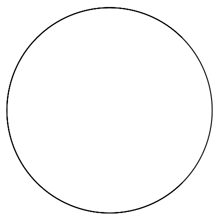
Spherical geometry



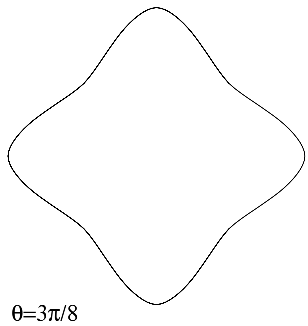
θ=0 [4-point scheme]



θ=π/8

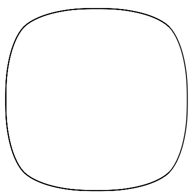


θ=π/4 [Perfect circle]

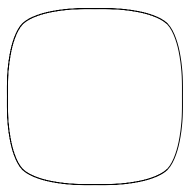


θ=3π/8

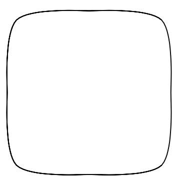
Hyperbolic geometry



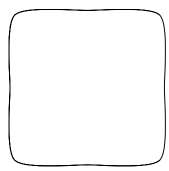
θ=0 [4-point scheme]



θ=0.5



θ=1



θ=1.5